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# Is ‘bosonic matter’ unstable in 2D?

**E B Manoukian and C Muthaporn**

School of Physics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand

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**Abstract**
An upper bound is derived for the exact ground-state energy in 2D, $E_N \leqslant -(me^4/2\hbar^2)(N^{3/2}/50\pi^2)$, of ‘bosonic matter’ consisting of $N$ positive and $N$ negative charges with Coulombic interactions. This is to be compared with the classic $N^{7/5}$ 3D-law of Dyson and gives rise to a more ‘violent’ collapse of such matter in 2D for large $N$. The derivation is based on a rigorous analysis which, in the process, controls the negative part of the Hamiltonian over its positive kinetic energy part and detailed estimates needed for counting trial wavefunctions of arbitrary states. A formal dimensional analysis in the style of Dyson alone shows, in arbitrary dimensions of space $d = 1, 2, \ldots,$ that $E_N \simeq -(me^4/2\hbar^2)C_d N^\rho$, $\rho = (d+4)/(d+2)$, where $C_d$ is a positive constant depending on $d$, consistent with our rigorous bound, and we are led to conjecture that ‘bosonic matter’ is unstable in all dimensions.

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## 1. Introduction and orientation

Over thirty years ago, Dyson [1] made a remarkable analysis in deriving an upper bound for the ground-state energy $E_N$ of the Hamiltonian

$$H' = \sum_{i=1}^{2N} \frac{p_i^2}{2m_i} + \sum_{i<j}^{2N} \frac{e^2 \varepsilon_i \varepsilon_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \tag{1}$$

consisting of $N$ positive and $N$ negative charges, where $\varepsilon_i = \pm 1$ in 3D giving rise to the famous $N^{7/5}$ law [1, 2] for bosons. The Dyson upper bound has now been improved [3] by over a factor of 30 and is given by [3]

$$E_N \leqslant -\left(\frac{me^4}{2\hbar^2}\right)\frac{N^{7/5}}{62\pi^4} \tag{2}$$

where $m$ is the smallest of the masses involved in (1). Such a power-law behaviour $N^\alpha$, with $\alpha > 1$, implies a collapse of bosonic matter since the formation of such matter consisting of $(2N + 2N)$ particles will be favourable over two separate systems brought into contact, each consisting of $(N+N)$ particles, and the energy released upon collapse, in the formation of the former system, being proportional to $[(2N)^{7/5} - 2(N)^{7/5}]$, will be overwhelmingly large for realistically large $N$, e.g., $N \sim 10^{23}$.

With the current interest in the physics of 2D and the connection of the spin and statistics e.g., [4–7], it is essential to investigate such bosonic systems, of central importance, in 2D. Our upper bound derivation for the ground-state energy in 2D is given by

$$E_N \leqslant -\left(\frac{me^4}{2\hbar^2}\right)\frac{N^{3/2}}{50\pi^2} \tag{3}$$

implying, in particular, collapse and, in general, a more 'violent' one for large $N$ than in 3D. Such collapsing matter may also be considered as collapsing planar matter sheets set side by side in 3D.

Although many papers have appeared recently on the stability of matter, e.g., [2, 8–11] and references therein, this paper is rather involved with the instability problem which is much more difficult [11, p 29]. The reason is that the problem of instability provides, in general, the necessary condition for the fermionic character of the electron for stability, while the analysis involved with the stability problem establishes the fermionic character as a sufficient condition for stability. Also as both signs of the charges are present in this work, the analysis becomes much more involved than that involved with only one sign of the charge, e.g. [8].

Since the kinetic energy operator is positive, we have for an arbitrary state $\Phi$, the bound $\langle\Phi|p_i^2|\Phi\rangle \leqslant (m_i/m)\langle\Phi|p_i^2|\Phi\rangle$, where $m$ is the smallest of the masses appearing in (1) (or it may even be taken to be smaller than that), it is sufficient for the purpose of obtaining an upper bound for the ground-state energy to consider instead of the Hamiltonian in (1), the following one:

$$H = \sum_{i=1}^{2N}\frac{p_i^2}{2m_i} + \sum_{i<j}^{2N}\frac{e^2\varepsilon_i\varepsilon_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}. \tag{4}$$

For a quantitative treatment, $m$, for example, may be taken to coincide with the mass of the electron.

In section 2, we summarize, for completeness, what is known [1, 3] of a general expression for an upper bound of $E_N$ in terms of the expectation values of a single-particle kinetic energy operator $-\hbar^2\nabla^2/2m$, and the Coulomb potential $e^2/|\boldsymbol{x} - \boldsymbol{x}'|$, with respect to single-particle trial wavefunctions. Several basic estimates needed in our final analysis for deriving the bound in (3), which develop a way of counting ordered quantum states in 2D, are established in section 3. These estimates of central importance are then used to derive our upper bound in section 4.

In estimating the ground-state energy of a system, trial functions, not necessarily coinciding with the exact ground-state wavefunction, lead to upper bounds to it from the very definition of a ground-state energy. The trial functions chosen in this work are quite suitable for making explicit estimates and lead to high localization of the particles for large $N$ in conformity with a dimensional analysis of $E_N$ given below.

As we will see in the following sections, the two-dimensional case may be treated rigorously and we have been interested in this analysis for several reasons. The immediate physical question arises: can 'bosonic matter' arrange itself in stable two-dimensional, so-called planar configurations? Our analysis shows that this cannot happen and such matter will not sustain itself in such configurations. Also the question arises as to whether the collapse is a characteristic of the dimensionality of space. A formal dimensional analysis in the style of Dyson alone given below, in arbitrary dimensions $d = 1, 2, \ldots,$ shows that

$$E_N \simeq -(me^4/2\hbar^2)C_d N^{(d+4)/(d+2)}$$

where $C_d$ is a positive constant depending on $d$, and this estimate is consistent with our rigorous bound in (3). According to this formal dimensional analysis, we are led to conjecture that 'bosonic matter ' is unstable in all dimensions. There has also been much interest in recent years in the physics of two dimensions, especially in condensed matter physics, and of the role of the spin and statistics theorem in such a lower dimension. It is well known that the latter theorem is tied up with the dimensionality of space, e.g. [4], and we concur with Dyson that 'bosonic matter', not being subject to stringent constrained statistics, is necessarily unstable, but as we now see this is also true in 2D. It is also an important theoretical question to investigate if the change of the dimensionality of space will change such matter from an 'implosive' to a 'stable' or to an 'explosive' phase. The formal dimensional analysis given indicates that this does not happen for all $d$. (Some of the present field theories speculate that at early stages of our universe the dimensionality of space was not necessarily coinciding with $d = 3$, and by a process which may be referred to as compactification of space, the present three-dimensional character of space arose upon the evolution and the cooling down of the universe.)

A preliminary rigorous variational analysis in deriving a lower bound for $E_N$, *vis à vis* our upper bound in (3), leads to singular expressions in our estimates due to the singular nature of the Coulomb potential. (We note, for example, in passing, that the Coulomb potential is locally square-integrable in 3D but not in 2D.) Also a preliminary analysis equivalent to deriving the bound in (3) for $d = 1$ and $d > 3$ seems to lead to formidable mathematical problems. For the former case, again, this is so because of the singular nature of the Coulomb potential. For the latter cases, the counting of trial wavefunctions for arbitrary states seems to require very extensive enumeration estimates. In a future report we hope to turn to all of these extensions which are beyond the scope of this paper.

A formal dimensional analysis of $E_N$ in the style of Dyson may be given, for completeness, for all space dimensions $d = 1, 2, \ldots,$ as reported above. To this end, let $L$ denote the overall extension of matter and let $\lambda$ denote the range of two-particle correlations. The total energy $E_N$ of the system may be formally written as

$$E_N = E_K^1 + E_K^2 + E_C$$

where $E_K^1$ is the kinetic energy associated with the overall system, $E_K^2$ being associated with the kinetic energy of the internal motion of short-range correlations as a cooperative effect of oppositely charged particles rushing to screen each other and $E_C$ being the Coulomb energy.

For $N$ positive charges and $N$ negative charges, an elementary dimensional analysis gives

$$E_K^1 \simeq 2N\left(\frac{\hbar^2}{2mL^2}\right) = \frac{N\hbar^2}{mL^2}.$$

Around each particle, a charged cloud with an overall opposite charge will arise, giving rise to an energy of the order $-e^2/\lambda$ for the interaction of each charge with its cloud, restricting only to short-range correlations due to the overall neutrality of matter considered at large. Also the self-energy of the cloud gives rise to an energy of the order $+e^2/2\lambda$. That is, as an order of magnitude estimate

$$E_C \simeq 2N\left(-\frac{e^2}{\lambda} + \frac{e^2}{2\lambda}\right) = -\frac{Ne^2}{\lambda}.$$

The charge cloud around each particle is produced by a cooperation of all particles within a volume $\sim(\lambda)^d$, with the number $\nu$ of these particles of the order $\nu \simeq N(\lambda/L)^d$. To change the charge of a cloud by one unit of charge, the probability density of each of the $\nu$ particles changes as

$$|\Psi|^2 \rightarrow |\Psi|^2 \pm \frac{1}{\nu}|\Psi|^2$$

as a particle moves from the surface of the cloud to its interior to produce the net excess or deficiency of one unit of charge within the cloud. For the relative change $|\delta\Psi/\Psi|$ of the wavefunctions of each of the $\nu$ particles, we may then take

$$|\delta\Psi/\Psi| \simeq \frac{1}{2\nu}$$

which upon carrying out a Taylor expansion in $\lambda$ gives the order of magnitude estimate

$$\lambda|\nabla\Psi/\Psi| \simeq 1/2\nu.$$

From this, we may infer that the kinetic energy per particle in each cloud, while forming the latter, is of the order

$$K \simeq \frac{\hbar^2}{8m}\frac{1}{(\nu\lambda)^2}.$$

An order of magnitude estimate then gives

$$E_K^2 \simeq (2N)\nu\frac{\hbar^2}{8m}\frac{1}{(\nu\lambda)^2} = \frac{\hbar^2}{4m}\frac{L^d}{\lambda^{d+2}}.$$

Upon minimizing $E_K^1 + E_K^2 = E_K$ over $L$, we obtain

$$L \sim \lambda N^{1/(2+d)}$$

and

$$E_K \simeq \frac{\hbar^2}{m\lambda^2}N^{d/(2+d)}A_d$$

where $A_d$ is some positive constant depending on $d$. Finally, upon minimizing

$$E_N = E_K - Ne^2/\lambda$$

over $\lambda$ gives $\lambda \sim (\hbar^2/me^2)N^{-2/(d+2)}$, and for $E_N$ the expression

$$E_N \simeq -(me^4/2\hbar^2)C_d N^{(d+4)/(2+d)}$$

where $C_d > 0$ and depends on $d$. It is interesting to note that optimizing $E_N$ over $L$ and $\lambda$, gives

$$L \sim (\hbar^2/me^2)N^{-1/(2+d)}$$

in conformity with our rigorous estimate in (43) with $\alpha = 1/4$ (see also (45)) for $d = 2$. We also note that $\nu \sim N^{2/(2+d)}$ which indicates that more particles participate, as a cooperative effect, in screening each other as $d$ decreases.

## 2. General upper bound expression of $E_N$

We set $z = (\boldsymbol{x}, \varepsilon)$ and introduce trial two-particle states [1]

$$G(z, z') = \sum_{\alpha=0}^{k} \lambda_\alpha c_\alpha(\varepsilon)c_\alpha(\varepsilon')\Psi_\alpha(\boldsymbol{x})\Psi_\alpha(\boldsymbol{x}') \tag{5}$$

where $\Psi_0(\boldsymbol{x}), \Psi_1(\boldsymbol{x}), \ldots, \Psi_k(\boldsymbol{x})$ are mutually orthonormal and the integer $k$ will be conveniently chosen later, with coefficients in (5) given by [1]

$$\lambda_0 = 1 \qquad \lambda_\alpha = -1/2 \qquad \alpha = 1, 2, \ldots, k \tag{6}$$

$$c_\alpha(\varepsilon) = \frac{1}{\sqrt{2}}\begin{cases} 1 & \alpha = 0 \\ \varepsilon & \alpha = 1, 2, \ldots, k. \end{cases} \tag{7}$$

One may then define a 2$N$-particle wavefunction as follows:

$$\Psi_{2N}(z_1, \dots, Z_{2N}) = \sum_{\pi} G(z(\pi_1), z(\pi_2)) \cdots G(z(\pi_{2N-1}), z(\pi_{2N})). \tag{8}$$

The sum is over all permutations $(\pi_1, \dots, \pi_{2N})$ of $(1, \dots, 2N)$. The wavefunction $\Psi_{2N}$ needs to be normalized. Since $\Psi_N/\|\Psi_N\|$ does not necessarily coincide with the ground-state function of $H$, the expectation value of $H$ with respect to $\Psi_N/\|\Psi_N\|$ cannot be less than the corresponding ground-state energy. That is, the expression in (8) can only provide an upper bound for the ground-state energy.

Derivation of a general upper bound expression for $E_N$ by carrying out, in the process, the expectation value of $H$ with respect to $\Psi_{2N}/\|\Psi_{2N}\|$ turns out to be very tedious [1, 3]. To give the general expression for this upper bound, we define the following expectation values with respect to single-particle trial wavefunctions $\Psi_\alpha(\boldsymbol{x})$:

$$T_\alpha = \int \mathrm{d}^2\boldsymbol{x}\, \Psi_\alpha^*(\boldsymbol{x}) \left(-\frac{\hbar^2\nabla^2}{2m}\right) \Psi_\alpha(\boldsymbol{x}) \qquad \alpha = 0, 1, \dots, k \tag{9}$$

$$I_{0_\alpha} = \int \mathrm{d}^2\boldsymbol{x}\, \mathrm{d}^2\boldsymbol{x}'\, \Psi_0^*(\boldsymbol{x})\Psi_\alpha^*(\boldsymbol{x}) \frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|} \Psi_0(\boldsymbol{x}')\Psi_\alpha(\boldsymbol{x}') \qquad \alpha = 1, \dots, k. \tag{10}$$

We then have the following general expression for the bound [3, equation (2.18)]:

$$E_N \leqslant \frac{1}{2}\sum_{\alpha=1}^{k} T_\alpha + \left(N - \frac{k}{3}\right) T_0 + \frac{1}{3}\left[N - (k-2)\frac{1}{2}\right]\left(-\sum_{\alpha=1}^{k} I_{0_\alpha}\right) \tag{11}$$

where $k < N$ and we note, in particular, that the coefficients of $T_0$ and $\left(-\sum_{\alpha=1}^{k} I_{0_\alpha}\right)$ are strictly positive.

## 3. Basic estimates

To derive the bound in (3), we need, in the process, to establish the bounds given, in turn, in (12), (18), (19) and (23).

To the above end, we consider the following construction. For each doublet $(n_1, n_2)$ of two natural numbers, we define a state specified by the tip of the vector $\boldsymbol{n} = (n_1, n_2)$. A non-trivial permutation of $(n_1, n_2)$ defines a different state. For example, (1, 2), (2, 1) define two distinct states satisfying, however, the constraint $n^2 = 5$.

For any such given allowed $n^2$ (a natural number), let $k$ denote the number of distinct states, excluding the state (1, 1), with the constraint that the length squared of each vector specifying such a state is less than or equal to $n^2$. This is the total number of states, excluding the state (1, 1), lying within, or falling on, a quarter of a circle of radius $n$ in the so-called first quadrant, i.e., for $n_1 \geqslant 1, n_2 \geqslant 1$.

Since by definition the state (1, 1) is excluded, the lowest possible value of $n^2$ is 5. For $n^2 = 5$, we have $k = 2$ corresponding to the states (1, 2), (2, 1). The next allowed value for $n^2$ is 8, with $k = 3$, corresponding to the states (1, 2), (2, 1), (2, 2), and so on for other values of $n^2 = 10, 13, 17, \dots$. We now establish the following.

**Proposition 1.** *For any allowed $n^2$, as defined above, we have the following inequality for the number of states $k$, also defined above, in relation to $n$:*

$$\frac{k}{n^2} \geqslant \sqrt{1 - \frac{1}{n^2}}\left(1 - \frac{3}{2n}\right) - \frac{1}{n} - \frac{1}{2} \tag{12}$$

*and the right-hand side of this inequality is strictly positive for allowed values of $n^2 \geqslant 29$.*

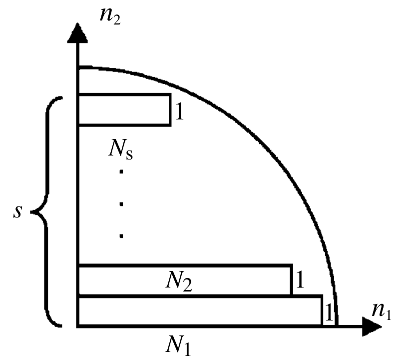


**Figure 1.** $s$ rectangles, each of unit height and bases of sizes $N_1, N_2, \ldots, N_s$, where the $N_i$ are positive integers, with $1 \leqslant N_s \leqslant \cdots \leqslant N_1$, are stacked on top of each other inside a quarter of a circle of radius $n$. Bounds are obtained on $N_1, N_2, \ldots, N_s$, such that the rectangles are within or just touch the circumference of a quarter of a circle.

To establish (12), $s$ rectangles, each of unit height and bases of sizes $N_1, N_2, \ldots, N_s$, where the $N_i$ are positive integers defined below with $N_1 \leqslant \cdots \leqslant N_s$, are stacked on top of each other as shown in figure 1 inside a quarter of a circle of radius $n$. Since the height of each rectangle is of one unit, we choose $N_1, \ldots, N_s$ to be the largest positive integers such that

$$N_1^2 + 1 \leqslant n^2, \ldots, N_s^2 + s^2 \leqslant n^2 \tag{13}$$

to make sure that the rectangles fall within or just touch the circumference of a quarter of the circle of radius $n$. That is, we take

$$N_1 \leqslant \sqrt{n^2 - 1} \leqslant N_1 + 1, \ldots, N_s \leqslant \sqrt{n^2 - s^2} \leqslant N_s + 1. \tag{14}$$

Also $N_s \geqslant 1$ requires that $1 \leqslant \sqrt{n^2 - s^2} \leqslant N_s + 1$. Hence $s$ is taken to be the largest positive integer such that

$$s \leqslant \sqrt{n^2 - 1} \leqslant s + 1. \tag{15}$$

Excluding the state (1, 1), the total number $k$ of states which lie within or are on the circumference of a quarter of the circle of radius $n$ clearly satisfies

$$k \geqslant \sum_{j=1}^{s} N_j - 1 \tag{16}$$

or

$$\begin{aligned} k &\geqslant \sum_{j=1}^{s} \left(\sqrt{n^2 - j^2} - 1\right) - 1 \\ &\geqslant \sum_{j=1}^{s} [(n - j) - 1] - 1 \\ &= (n - 1)s - \frac{s(s + 1)}{2} - 1. \end{aligned} \tag{17}$$

Upon using both inequalities in (15), as the case may be, (17) leads to (12).

Many estimates of the sort in (12), involving corrections, are available in the literature. A classic example of this is filling a sphere with smaller spheres (the so-called Swiss-cheese theorem) see, e.g., [12, 13]. The estimate in (12), as it stands, is not, however, what is ultimately needed. What we need is a more involved one which allows us to count $(k - k')$

states corresponding to two consecutive $n^2 > n'^2$ values which is of central importance in deriving the upper bound for the ground-state energy in section 4. This estimate is given in proposition 3. We first establish the following result.

**Proposition 2.** *Let* $n'^2 < n^2$ *be* consecutive *allowed* $n^2$ *values, then*

$$n - n' \leqslant 1 + \frac{1}{n'} \tag{18}$$

*and*

$$n\left[1 - \frac{1}{n'}\left(1 + \frac{1}{n'}\right)\right] \leqslant n'. \tag{19}$$

To derive (18), note that although $n'^2$ is a natural number, $n'$ is not necessarily so. Accordingly, let $n'_0$ be the largest positive integer such that $n'_0 \leqslant n$. That is, we may write

$$n' = n'_0 + x \qquad 0 \leqslant x < 1. \tag{20}$$

Consider the state specified by the vector $\boldsymbol{n}'' = (n'_0 + 1, 1)$. Clearly, $n'' > n'$. Since $n^2, n'^2$ are consecutive with $n^2 > n'^2$, it follows that $n'' \geqslant n$. Now

$$2n'(n'' - n') \leqslant n''^2 - n'^2 = 2n'_0(1 - x) + (2 - x^2)$$

which from $n \leqslant n''$, $0 \leqslant x < 1$, leads to (18).

Upon rewriting (18) as

$$n - \left(1 + \frac{1}{n'}\right) \leqslant n' \tag{21}$$

and using the fact that $n > n'$, (19) follows.

For any consecutive $n^2 > n'^2$, we label the $(k - k')$ states, specified by those vectors *all* of length squared equal precisely to $n^2$ in an arbitrary order, as $\alpha = k' + 1, k' + 2, \ldots, k$. Let

$$C(n') = \sqrt{1 - \frac{1}{n'^2}}\left(1 - \frac{3}{2n'}\right) - \frac{1}{n'} - \frac{1}{2} \tag{22}$$

which coincides with the right-hand side expression in (12) when $n$ is replaced by $n'$. We then have the following important result.

**Proposition 3.**

$$\alpha \geqslant n^2\left[1 - \frac{1}{n'}\left(1 + \frac{1}{n'}\right)\right]^2 C(n') \tag{23}$$

*valid for* $n'^2 \geqslant 29$.

This inequality follows from that in (12) which leads, in the process, to

$$k > \cdots > (k' + 1) > k' \geqslant n'^2 C(n') \tag{24}$$

and the one in (19). The constraint $n'^2 \geqslant 29$ just ensures the positivity of $C(n')$. (For the state specified by the vector $\boldsymbol{n}' = (2, 5)$, for example, $n'^2 = 29$.)

For $n^2 \leqslant 109$, an elementary computer analysis shows that $\alpha \geqslant n^2/5$. On the other hand, for $n^2 \geqslant 109$, we may use our explicit inequality in (23) (valid for $n^2$ up to infinity!) to conclude that $\alpha > n^2/5$. That is, for all allowed $n^2$,

$$\alpha \geqslant n^2/5. \tag{25}$$

## 4. Derivation of the upper bound

For orthonormal trial functions, we choose the Dyson ones [1]

$$\Phi_n(\boldsymbol{x}) = \frac{2}{L}\sin\frac{n_1\pi x_1}{L}\sin\frac{n_2\pi x_2}{L} \tag{26}$$

for $0 < x_i < L$, and vanishing outside this interval. We label the states as $\alpha = 0$ for $\boldsymbol{n}_0 = (1, 1)$ and for $\alpha \geqslant 1$, $\alpha = 1, 2, 3$ for $\boldsymbol{n} = (1, 2), (2, 1), (2, 2)$, respectively and so on.

With our effort in deriving the bound given below, we have found the Dyson trial functions most suitable for the problem at hand for the following reasons. (1) We need an orthonormal set of functions, defined on a bounded interval, for each $x_i$, vanishing at its endpoints with the length of the interval, chosen optimally, becoming smaller and smaller as $N$ increases, implying the localization of the particles and eventual collapse for large $N$. (2) The trial orthonormal functions in (26) are simple enough to make explicit sharp analytical estimates as is seen below. (3) We have tried other orthonormal trial functions, such as the Hermite functions, with an arbitrary scale parameter, and in all cases analysed that the negative interaction part becomes very small compared to the kinetic energy part for large $n$, and hence are not appropriate as trial functions. In particular, the normalization constant in (26) is independent of $n$ unlike the situation, for example, with Hermite functions. (4) The trial functions in (26) overlap, which is what is needed for the interaction term in (28) to be non-vanishing, and a choice of non-overlapping orthonormal states defined on sub-intervals of $(0, L)$, for each $x_i$, for example, is not useful.

Let

$$T^{(\boldsymbol{n})} = \left\langle \Phi_{\boldsymbol{n}} \left| -\frac{\hbar^2\nabla^2}{2m} \right| \Phi_{\boldsymbol{n}} \right\rangle \tag{27}$$

$$I^{(\boldsymbol{n})} = \int \mathrm{d}^2\boldsymbol{x}\,\mathrm{d}^2\boldsymbol{x}'\,\Phi_{\boldsymbol{n}_0}(\boldsymbol{x})\Phi_{\boldsymbol{n}}(\boldsymbol{x})\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\Phi_{\boldsymbol{n}_0}(\boldsymbol{x}')\Phi_{\boldsymbol{n}}(\boldsymbol{x}) \tag{28}$$

where we note that the functions in (26) are real. The evaluation of the integral in (27) is straightforward and gives

$$T^{(n)} = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2} \tag{29}$$

$$T^{(n_0)} = \frac{\hbar^2}{m}\frac{\pi^2}{L^2} \tag{30}$$

where $n_0^2 = 2$. The evaluation of (28) is more involved. To obtain an appropriate bound for (28), we define

$$F_{nm}(\boldsymbol{k}) = \int \mathrm{d}^2\boldsymbol{x}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}[\Phi_{n}(\boldsymbol{x})\Phi_{m}(\boldsymbol{x})] \tag{31}$$

and hence

$$\int \mathrm{d}^2\boldsymbol{x}|\Phi_n(\boldsymbol{x})\Phi_m(\boldsymbol{x})|^2 = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\sqrt{|\boldsymbol{k}|}|F_{nm}(\boldsymbol{k})|\frac{|F_{nm}(\boldsymbol{k})|}{\sqrt{|\boldsymbol{k}|}}. \tag{32}$$

From the elementary Schwarz inequality, we also have

$$\left(\int \mathrm{d}^2\boldsymbol{x}|\Phi_n(\boldsymbol{x})\Phi_m(\boldsymbol{x})|^2\right)^2 \leqslant \left(\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}|\boldsymbol{k}||F_{nm}(\boldsymbol{k})|^2\right)\left(\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\frac{|F_{nm}(\boldsymbol{k})|^2}{|\boldsymbol{k}|}\right) \tag{33}$$

or

$$\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\frac{|F_{nm}(\boldsymbol{k})|^2}{|\boldsymbol{k}|} \geqslant \frac{\left(\int \mathrm{d}^2\boldsymbol{x}|\Phi_n(\boldsymbol{x})\Phi_m(\boldsymbol{x})|^2\right)^2}{\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}|\boldsymbol{k}||F_{nm}(\boldsymbol{k})|^2}. \tag{34}$$

Also the integral

$$\int \mathrm{d}^2\boldsymbol{x}\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}}{|\boldsymbol{x}|} = \frac{2\pi}{|\boldsymbol{k}|} \tag{35}$$

gives

$$I^{(n)} = (2\pi e^2)\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\frac{|F_{nm}(\boldsymbol{k})|^2}{|\boldsymbol{k}|}. \tag{36}$$

On the other hand,

$$\begin{aligned}\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}|\boldsymbol{k}||F_{nm}(\boldsymbol{k})|^2 &= \int \mathrm{d}^2\boldsymbol{x}\,\Phi_n(\boldsymbol{x})\Phi_m(\boldsymbol{x})\sqrt{-\nabla^2}\Phi_n(\boldsymbol{x})\Phi_m(\boldsymbol{x}) \\ &= \sqrt{(n^2+m^2)\frac{\pi^2}{L^2}}\int \mathrm{d}^2\boldsymbol{x}|\Phi_n(x)\Phi_m(\boldsymbol{x})|^2.\end{aligned} \tag{37}$$

The inequality (34) then gives for (28)

$$I^{(n)} \geqslant \frac{2e^2}{L\sqrt{n^2+2}}. \tag{38}$$

From the basic estimate in (25), we then obtain

$$T_\alpha \leqslant \frac{5\hbar^2}{2m}\frac{\pi^2}{L^2}\alpha \tag{39}$$

$$I_{0\alpha} \geqslant \frac{2e^2}{\sqrt{7k}L} \tag{40}$$

with $T_\alpha$, $I_{0\alpha}$ defined in (9), (10), respectively, and hence

$$\sum_{\alpha=1}^{k} T_\alpha \leqslant \frac{5}{2}\frac{\hbar^2\pi^2}{mL^2}\frac{k(k+1)}{2} \tag{41}$$

$$\sum_{\alpha=1}^{k}(-I_{0\alpha}) \leqslant -\frac{2e^2\sqrt{k}}{\sqrt{7}L}. \tag{42}$$

The latter two inequalities are needed in our upper bound in (11).

Upon setting

$$\frac{1}{L} = \frac{me^2}{\hbar^2}A\frac{N^\alpha}{\pi^2} \tag{43}$$

where $A$ and $\alpha$ are optimally determined, we obtain from (11)

$$E_N \leqslant \frac{me^4}{2\hbar^2}\left(\frac{N^{1+2\alpha}}{\pi^2}\right)\left\{A^2\left[2\left(1-\frac{k}{3N}\right)+\frac{5}{4}\frac{k(k+1)}{N}\right]-\frac{4}{3}\left(1-\frac{(k-2)}{2N}\right)\frac{\sqrt{k}}{\sqrt{7}}\frac{A}{N^\alpha}\right\}. \tag{44}$$

Optimally, we choose

$$A = \frac{2}{3}\frac{\left(1-\frac{(k-2)}{2N}\right)\sqrt{\frac{k}{7}\frac{1}{N^\alpha}}}{\left[2\left(1-\frac{k}{3N}\right)+\frac{5}{4}\frac{k(k+1)}{N}\right]} \tag{45}$$

$\alpha = 1/4$, and $k$ as the largest positive integer $k < N$, such that

$$k \leqslant \left(\tfrac{8}{5}N\right)^{1/2} \leqslant k+1 \tag{46}$$

for large $N$. The inequality in (44) then leads to

$$E_N \leqslant -\frac{me^4}{2\hbar^2}\frac{N^{3/2}}{50\pi^2} \tag{47}$$

as stated in (3). The bound in (47) will be useful if, for example, the upper bound in (47) is less than the ground-state energy of $N$ isolated boson-equivalents of positronium atoms in 2D, that is, if

$$-\frac{N^{3/2}}{50\pi^2} < -\frac{N}{2} \tag{48}$$

where the $1/2$ factor on the right-hand side arises as a result of the reduced mass of an atom. Equation (48) gives the constraint $N > (25\pi^2)^2 \simeq 6.1 \times 10^4$ consistent with a large $N$ in our analysis. Physically, however, one is interested in much larger $N$, e.g., $N \sim 10^{23}$.

## 5. Conclusion

A rigorous upper bound for the exact ground-state energy was derived in (47) in 2D for 'bosonic matter' consisting of $N$ positive and $N$ negative charges with Coulombic interactions giving rise to a $N^{3/2}$ 2D-law for bosons. Compared to the classic $N^{7/5}$ 3D-law of Dyson, this gives rise to a more 'violent' collapse of such matter in 2D for large $N$. In particular, 'bosonic matter' cannot arrange itself, in the bulk, in stable two-dimensional planar configurations. A formal dimensional analysis of the ground-state energy gives rise to a $N^{(d+4)/(d+2)}$ law for bosons in $d = 1, 2, \ldots,$ dimensional spaces, consistent with our rigorous bound, and led us to conjecture that 'bosonic matter' is unstable in all dimensions. The extension of the rigorous analysis carried out here to the cases $d = 1$ and $d > 3$ seems to lead to formidable mathematical problems, as well as a derivation of a low bound to $E_N$ *vis à vis* our bound in (47), and are beyond the scope of this paper. We hope to come back to such extensions in a future report.

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